Q-capita on the arbelos

Hein van Winkel
hjvwinkel at gmail.com

July 7, 2015

Abstract

Some Q-configurations related to the Arbelos. I have let me inspire for this article by Harold P. Boas: 'Reflections on the Arbelos' ; The Mathematical association of America (Monthly 113 March 2006). In the first three sections I present some basic information with nice Q–configurations on the arbelos. In the fourth sections the chain of Pappos with some thoughts of Jakob Steiner leads to formulas for π.

Contents

1 Circumference and area of the arbelos

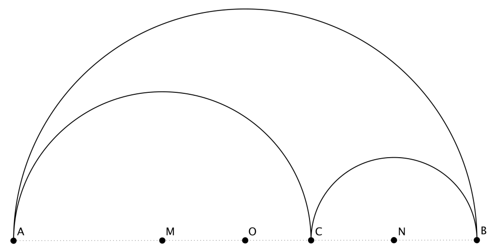


Figure 1: The arbelos

The arbelos consists of three halfcircles, tangent to each other. The point C in figure 1 lies on the linesegment AB. To get Q-configurations w'll choose for an half unit-circle and rationals for the linesegments AC and BC.

Some notations and other that we will use in this paper.
The diameter of the half-circle AB is 2, the midpoint O.
The diameters of the half-cicles AC and BC are respectively $2a$ and $2b$ and their midpoints M and N. Remark: $a+b=1$.
The ratio $\frac{AC}{BC}$ will be $\rho = \frac{a}{b} = \frac{a}{1-a} = \frac{1-b}{b}$.
Coordinates of the points on the diameter AB in a, b and ρ :

$$
\begin{aligned}
&A(-1,0), B(1,0), O(0,0)\\
&M(-1+a,0) = (-b,0) = \left(-\frac{1}{\rho+1}, 0\right)\\
&N(a,0) = (1-b,0) = \left(\frac{\rho}{\rho+1}, 0\right)\\
&C(-1+2a,0) = (1-2b,0) = \left(\frac{\rho-1}{\rho+1}, 0\right)
\end{aligned}
$$

Circumference of the arbelos equals $arcAB + arcBC + arcAC = \frac{1}{2}(2\pi + 2\pi a + 2\pi b) = \pi(1+a+b) = 2\pi$
Area of the arbelos $\frac{1}{2}(\pi - \pi a^2 - \pi b^2) = \frac{1}{2}(\pi(a+b)^2 - \pi a^2 - \pi b^2) = \pi ab$

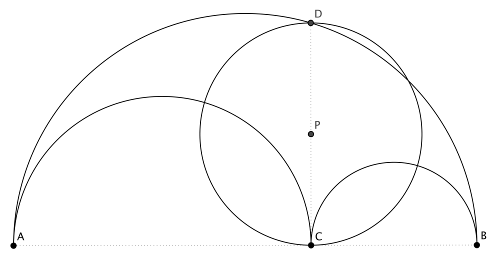


Figure 2: An area property of the arbelos

The perpendicular of AB in C intersects the arbelos in the point D.
Proposition.
The area of the arbelos equals the area of the circle with diameter CD.
Proof.
Euclides gives $CD^2 = AC \cdot CB$. And so it follows, that Area of circle with diameter CD equals
$\pi \cdot \frac{1}{4}CD^2 = \pi \cdot \frac{1}{4}AC \cdot CB = \pi \cdot \frac{1}{4} \cdot 2a \cdot 2b = \pi ab$.
We just found that this is the area of the arbelos.

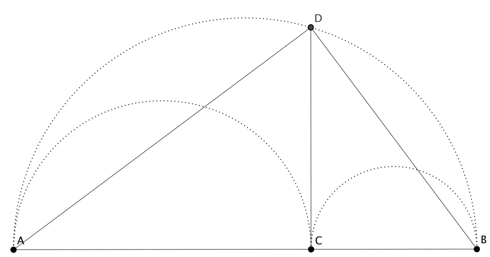


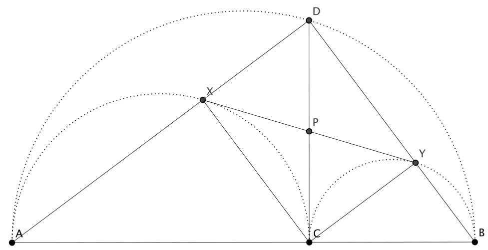
Figure 3: $ABCD$ as Q-configuration

Now look at the figure of the right-angled $\triangle ABD$ and let $\angle DAB = \alpha$ and $AB = 2$. Then we get:
the linesegments $AD = 2\cos\alpha$, $AC = 2\cos^2\alpha$, $BD = 2\sin\alpha$, $BC = 2\sin^2\alpha$, $CD = 2\sin\alpha\cos\alpha$
and the areas $\triangle ACD = 2\sin\alpha\cos^3\alpha$, $\triangle BCD = 2\sin^3\alpha\cos\alpha$.
Remark.
$a = \cos^2\alpha$ and $b = \sin^2\alpha$ We have a Q-configuration for quadratic rationals a and b.

In the following table a summery of the formulas, expressed in α, a, and ρ with the fifth column for $\cos^2\alpha = a = \frac{rho}{rho+1} = \frac{16}{25}$ and the last column scaled up with a factor $\frac{25}{2}$ to get the smallest Q-configuration.

AB	2	2	2	2	25
AD	$2\cos\alpha$	$2\sqrt{a}$	$2\sqrt{\frac{\rho}{\rho+1}}$	$\frac{8}{5}$	20
AC	$2\cos^2\alpha$	$2a$	$\frac{2\rho}{\rho+1}$	$\frac{32}{25}$	16
BD	$2\sin\alpha$	$2\sqrt{1-a}$	$2\sqrt{\frac{1}{\rho+1}}$	$\frac{6}{5}$	15
BC	$2\sin^2\alpha$	$2(1-a)$	$\frac{2}{\rho+1}$	$\frac{18}{25}$	9
CD	$2\sin\alpha\cos\alpha$	$2\sqrt{a(1-a)}$	$\frac{2}{\rho+1}\sqrt{\rho}$	$\frac{24}{25}$	12
ACD	$2\sin\alpha\cos^3\alpha$	$2a\sqrt{a(1-a)}$	$\frac{4\rho}{(\rho+1)^2}\sqrt{\rho}$	$\frac{384}{625}$	96
BCD	$2\sin^3\alpha\cos\alpha$	$2(1-a)\sqrt{a(1-a)}$	$\frac{4}{(\rho+1)^2}\sqrt{\rho}$	$\frac{216}{625}$	54

2 The inner tangent XY of the arbelos



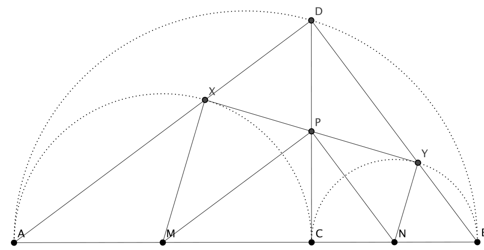


Figure 4: Q-configuration with tangent XY

DA and DB meet arcAC and arcBC respectively in the points X and Y. Then $CYDX$ is a rectangle, whose diagonals meet in P. Now $PX = PY = PC$ and thus XY is tangent to the arcs AC and BC. To find the formulas of the following table is easy now.

AB	2	2	2	2	125
AC	$2\cos^2\alpha$	$2a$	$\frac{2\rho}{\rho+1}$	$\frac{32}{25}$	80
AX	$2\cos^3\alpha$	$2\left(\sqrt{a}\right)^3$	$2\left(\sqrt{\frac{\rho}{\rho+1}}\right)^3$	$\frac{128}{125}$	64
BC	$2\sin^2\alpha$	$2(1-a)$	$\frac{2}{\rho+1}$	$\frac{18}{25}$	45
BY	$2\sin^3\alpha$	$2\left(\sqrt{1-a}\right)^3$	$2\left(\sqrt{\frac{1}{\rho+1}}\right)^3$	$\frac{54}{125}$	27
CP	$\sin\alpha\cos\alpha$	$\sqrt{a(1-a)}$	$\frac{1}{\rho+1}\sqrt{\rho}$	$\frac{12}{25}$	30
CX	$2\sin\alpha\cos^2\alpha$	$2\sqrt{a(1-a)}$	$\frac{2\rho}{\rho+1}\sqrt{\frac{1}{\rho+1}}$	$\frac{96}{125}$	48
CY	$2\sin^2\alpha\cos\alpha$	$2a\sqrt{a(1-a)}$	$\frac{2}{\rho+1}\sqrt{\frac{\rho}{\rho+1}}$	$\frac{72}{125}$	36
ACX	$2\sin\alpha\cos^5\alpha$	$2a^2\sqrt{a(1-a)}$	$\frac{2\rho^2}{(\rho+1)^3}\sqrt{\rho}$	$\frac{6144}{15625}$	1536
BCY	$2\sin^5\alpha\cos\alpha$	$2(1-a)^2\sqrt{a(1-a)}$	$\frac{2}{(\rho+1)^3}\sqrt{\rho}$	$\frac{1944}{15625}$	486
DPX	$\sin^3\alpha\cos^3\alpha$	$\sqrt{(a(1-a))^3}$	$\frac{\sqrt{\rho^3}}{(\rho+1)^3}$	$\frac{1728}{15625}$	432

As in the preceeding section formula's in α, a and ρ and in the last column upscaled to the smallest Q-configuration for the left figure. See next page for an additional table for the right figure.

$MA = MC = MX$	$\cos^2\alpha$	a	$\frac{\rho}{\rho+1}$	$\frac{16}{25}$	40
$NB = NC = NY$	$\sin^2\alpha$	$(1-a)$	$\frac{1}{\rho+1}$	$\frac{9}{25}$	22,5
PM	$\cos\alpha$	$\sqrt{a}$	$\sqrt{\frac{\rho}{\rho+1}}$	$\frac{4}{5}$	50
PN	$\sin\alpha$	$\sqrt{1-a}$	$\sqrt{\frac{1}{\rho+1}}$	$\frac{3}{5}$	37,5
$CPM = PXM$	$\frac{1}{2}\sin\alpha\cos^3\alpha$	$\frac{1}{2}a\sqrt{a(1-a)}$	$\frac{\sqrt{\rho^3}}{2(\rho+1)^2}$	$\frac{96}{625}$	600
$CPN = YPN$	$\frac{1}{2}\sin^3\alpha\cos\alpha$	$\frac{1}{2}(1-a)\sqrt{a(1-a)}$	$\frac{\sqrt{\rho}}{2(\rho+1)^2}$	$\frac{54}{625}$	337,5
AXM	$\sin\alpha\cos^5\alpha$	$a^2\sqrt{a(1-a)}$	$\frac{\rho^2}{(\rho+1)^3}\sqrt{\rho}$	$\frac{3072}{15625}$	768
BNY	$\sin^5\alpha\cos\alpha$	$(1-a)^2\sqrt{a(1-a)}$	$\frac{1}{(\rho+1)^3}\sqrt{\rho}$	$\frac{972}{15625}$	342

For the smallest Q-configuration there must be an extra upscaling with factor 2 for the linesegments.

3 The twin circles.

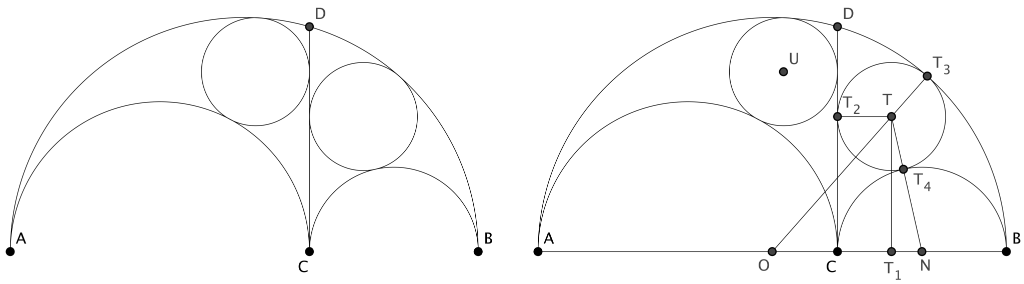


Figure 5: The twin circles

Archimedes found the twin-circles, tangent at the tangent CD and two of the other half-circles. In almost every paper about circles in the arbelos you can find the property that the twin circles are congruent.
We look at this property and will see that $\triangle MOU$ and $\triangle TON$ are Q-triangles.

Let r be the radius of the circle with center T. Then we have:
$TT_1^2 = OT^2 - OT_1^2 = TN^2 - NT_1^2 \Leftrightarrow$
$(1-r)^2 - (1-2b+r)^2 = (b+r)^2 - (b-r)^2 \Leftrightarrow r = b(1-b) = ab$
For the left one of the twins we get:
$UU_1^2 = OU^2 - OU_1^2 = UM^2 - MU_1^2 \Leftrightarrow$
$(1-r)^2 - (2a-1-r)^2 = (a+r)^2 - (a-r)^2 \Leftrightarrow r = a(1-a) = ab$

Proposition.
The circumcenters of a twincircle, and the halfcircles tangent to it form a Q-triangle, if the lengths of the radii of the halfcircles on AC and BC are rational.
Proof.
Let us now look for $\triangle TON$.
$TT_1^2 = (b+r)^2 - (b-r)^2 = 4br = 4ab^2$ gives $TT_1 = 2b\sqrt{a}$.
$OT_1 = 1 - 2b + r = 1 - 2b + b - b^2 = 1 - b - b^2$
$T(1-b-b^2, 2b\sqrt{1-b})$, $N(1-b, 0)$ and $O(0,0)$ are the vertices,
$ON = 1-b$, $NT = b(2-b)$ and $OT = 1 - b + b^2$ are the sidelengths
and Area$(\triangle TON) = b(1-b)\sqrt{1-b}$.
For the left $\triangle MOU$ we get in the same way:

$UU_1^2 = MU^2 - MU_1^2 = (a+r)^2 - (a-r)^2 = 4ar = 4a^2b \Rightarrow UU_1 = 2a\sqrt{1-a}$,
$U(-1+a+a^2, 2a\sqrt{1-a})$, $M(-1+a, 0)$ and $O(0,0)$ are the vertices,
$OM = -1+a$, $MU = a(2-a)$ and $OU = 1-a+a^2$ are the sidelengths
and Area$(\triangle MOU) = a(1-a)\sqrt{1-a}$.

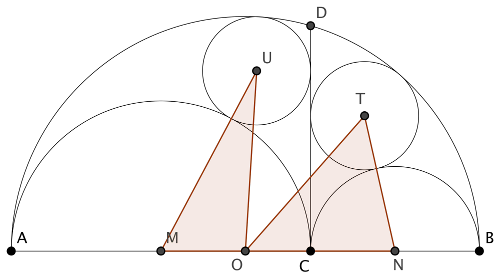


Figure 6: Twin circles and Q-triangles

Remark.
The length of TU is not rational.
Proof.
$TU^2 = (UU_1 - TT_1)^2 + (2r)^2 = (2a\sqrt{b} - 2b\sqrt{a})^2 + 4ab = 4ab(a - 2\sqrt{ab} + b + 1) = 8ab(1+\sqrt{ab}) = 8\cos^2\alpha\sin^2\alpha(1+\sqrt{\cos^2\alpha\sin^2\alpha})$
Suppose TU^2 be a rational square. Then we have
$2(1+\cos\alpha\sin\alpha)$ is a rational square.
$\Rightarrow 2\frac{q^2+pr}{q^2}$ must be a rational square with $\frac{p}{q} = \cos\alpha$ and $\frac{r}{q} = \sin\alpha$ and $(p,q,r) = 1$.
$\Rightarrow q^2 + pr = p^2 + r^2 + pr$ must be even.
This contradicts $(p,q,r) = 1$.

4 Pappos chains and formulas for π

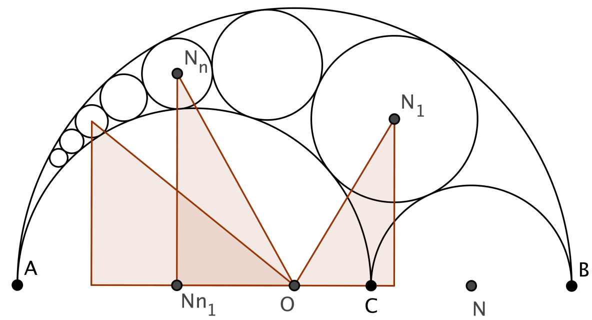


Figure 7: Pythagorean triangles and the Pappos chain

In the arbelos exists a chain of circles, named after the greek mathematician Pappos. The first circle will be the circle, which is tangent to the three arbelos-half-circles. It has midpoint N_1 and diameter $d_1 = 2r_1$. Let the n-th circle have midpoint N_n and diameter $d_n = 2r_n$. Let N_{n_1} be the pedal of the perpendicular through N_n on AB.
Pappos has shown, that $N_nN_{n_1} = nd_n$.
Steiner has shown by inversion with center A and inversioncircle perpendicular to the n-th circle of the chain, that $\frac{AN_{n_1}}{d_n} = \frac{AN}{BC} = \frac{AC+\frac{1}{2}BC}{BC} = \frac{AC}{BC} + \frac{1}{2} = \rho + \frac{1}{2}$. With $\rho = \frac{AC}{BC}$ as defined in section 1.

Cartesian coordinates and some computing give $N_n(x_n, y_n)$ with
– Steiner: $x_n = AN_{n_1} - 1 = d_n(\rho + \frac{1}{2}) - 1 = r_n(1 + 2\rho) - 1$
– Pappos: $y_n = 2nr_n$
– $ON_n = 1 - r_n$

We now calculate with Pythagoras:

$$
\begin{aligned}
&(r_n(1+2\rho)-1)^2 + (2nr_n)^2 = (1-r_n)^2 \\
&\Leftrightarrow (1+2\rho)^2 r_n^2 - 2(1+2\rho)r_n + 1 + 4n^2r_n^2 = r_n^2 - 2r_n + 1 \\
&\Leftrightarrow (4n^2 - 1 + (1+2\rho)^2)r_n = \rho \Leftrightarrow r_n = \frac{\rho}{(n^2+\rho+\rho^2)}
\end{aligned}
$$

Proposition (a).
When $\rho \in \mathbb{Q}$ then the points $\{N_n\}$ has rational coordinates and the polygon $ANN_1N_2N_3....$, devided in the triangles $ONN_1, ON_1N_2, ..., ON_{n-1}N_n, ...$ is a Q-configuration.
Proof.
For $n > 1$ we have:

$$x_n = r_n(1+2\rho) - 1 = \frac{\rho(1+2\rho)}{n^2+\rho+\rho^2} - 1 = \frac{\rho(1+2\rho) - (n^2+\rho+\rho^2)}{n^2+\rho+\rho^2} = \frac{-(n^2-\rho^2)}{n^2+\rho+\rho^2}$$
$$y_n = 2nr_n = \frac{2\rho n}{n^2+\rho+\rho^2}$$
$$ON_n = 1 - r_n = 1 - \frac{\rho}{n^2+\rho+\rho^2} = \frac{n^2+\rho^2}{n^2+\rho+\rho^2}$$
$$N_nN_{n-1} = r_n + r_{n-1} = \frac{\rho}{n^2+\rho+\rho^2} + \frac{\rho}{(n-1)^2+\rho+\rho^2}$$

For the area of $\triangle ON_nN_{n-1}$ we find

$$\begin{aligned} A_n = A_{ON_nN_{n-1}} &= \frac{1}{2}(x_{n-1}y_n - x_ny_{n-1}) \\ &= \frac{1}{2}\left(\frac{-((n-1)^2-\rho^2)}{(n-1)^2+\rho+\rho^2}\frac{2\rho n}{n^2+\rho+\rho^2} - \frac{-(n^2-\rho^2)}{n^2+\rho+\rho^2)}\frac{2\rho(n-1)}{(n-1)^2+\rho+\rho^2}\right) \\ &= \frac{\rho((n^2-\rho^2)(n-1) - ((n-1)^2-\rho^2)n)}{((n-1)^2+\rho+\rho^2)(n^2+\rho+\rho^2)} \\ &= \frac{\rho(n^2-n+\rho^2)}{((n-1)^2+\rho+\rho^2)(n^2+\rho+\rho^2)}. \end{aligned}$$

Corollary.
When we fix ρ then the formule gives for the area of the polygon $ANN_1N_2N_3....$

$$\sum_{i=1}^{\infty} \frac{\rho(i^2-i+\rho^2)}{((i-1)^2+\rho+\rho^2)(i^2+\rho+\rho^2)}$$

And this gives a very slow converging limit

$$\lim_{\rho\to\infty}\left(\sum_{i=1}^{\infty} \frac{\rho(i^2-i+\rho^2)}{((i-1)^2+\rho+\rho^2)(i^2+\rho+\rho^2)}\right) = \frac{1}{2}\pi \qquad (1)$$

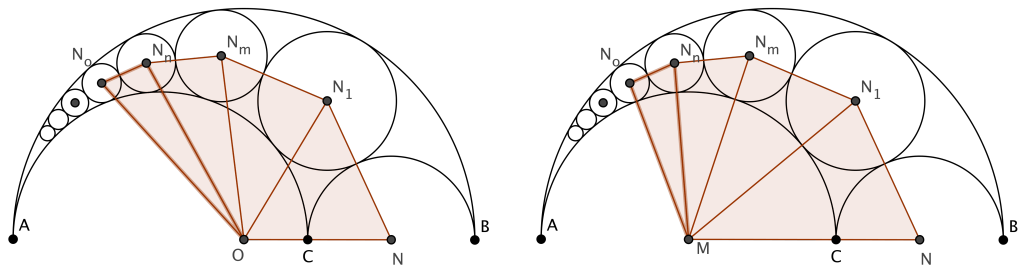


Figure 8: Q-polygon and formula for π

Steiner mentioned another row of pythagorean triangles with vertex M in stead of O. This leads to

Proposition (b).
When $\rho \in \mathbb{Q}$ then the points $\{N_n\}$ has rational coordinates and the polygon $ANN_1N_2N_3....$, devided in the triangles $MNN_1, MN_1N_2, ..., MN_{n-1}N_n, ...$ is a Q-configuration.
Proof.
Let B_n be the area of $\triangle MN_nN_{n-1}$. As in the proof of proposition (a) we find:

$$x_n = r_n(1+2\rho) - 1 = \frac{\rho(1+2\rho)}{n^2+\rho+\rho^2} - 1 = \frac{\rho(1+2\rho) - (n^2+\rho+\rho^2)}{n^2+\rho+\rho^2} = \frac{-(n^2-\rho^2)}{n^2+\rho+\rho^2}$$

$$y_n = 2nr_n = \frac{2\rho n}{n^2+\rho+\rho^2} = \frac{2\rho n}{n^2+\rho+\rho^2}$$

$$MN_n = a + r_n = \frac{\rho}{\rho+1} + \frac{\rho}{n^2+\rho+\rho^2} = \frac{\rho(n^2+(\rho+1)^2)}{(\rho+1)(n^2+\rho+\rho^2)}$$

$$N_nN_{n-1} = r_n + r_{n-1} = \frac{\rho}{n^2+\rho+\rho^2} + \frac{\rho}{(n-1)^2+\rho+\rho^2}$$

$$B_n = A_{MN_nN_{n-1}} = \frac{1}{2}((x_{n-1}+b)y_n - (x_n+1-a)y_{n-1})$$

$$= \frac{1}{2}\left(\left(\frac{-((n-1)^2-\rho^2)}{(n-1)^2+\rho+\rho^2} + \frac{1}{\rho+1}\right)\frac{2\rho n}{n^2+\rho+\rho^2} - \left(\frac{-(n^2-\rho^2)}{n^2+\rho+\rho^2} + \frac{1}{\rho+1}\right)\frac{2\rho(n-1)}{(n-1)^2+\rho+\rho^2}\right)$$

$$= \frac{\rho^2(n^2-n+(\rho+1)^2)}{(\rho+1)(n^2+\rho+\rho^2)((n-1)^2+\rho+\rho^2)}$$

And we have another very slow converging limit

$$\lim_{\rho\to\infty}\left(\sum_{i=1}^{\infty}\frac{\rho^2(i^2-i+(\rho+1)^2)}{(\rho+1)(i^2+\rho+\rho^2)((i-1)^2+\rho+\rho^2)}\right)=\frac{1}{2}\pi \tag{2}$$

To give an impression of the very slow speed of convergence of this formula, I used SAGE to compute with $\rho = 2500$ the sum of the first 500000 terms of the series. An approximation of this sum, corrected for the half ellips with long axis AN gives 3.13284330082914.
Someone, who loves to see very large numbers, can see this uncorrected sum at http://www.duizendknoop.com/b/pi-arb.pdf

We end this section with a third, less complicated formula for π. The broken line NN_1N_2...... has length

$$L = NN_1 + N_1N_2 + ... = (b + r_1) + (r_1 + r_2) + ... = b + 2r_1 + 2r_2 + ...$$

And so

$$\lim_{\rho\to\infty}\left(\frac{1}{\rho+1}+2\rho\sum_{i=1}^{\infty}\frac{1}{i^2+\rho+\rho^2}\right)=\pi \tag{3}$$